Theory Qual Spring 2017

Please answer 3 of the 4 problems given. If you worked on more than 3 problems, make sure which 3 you would like us to grade. Only 3 problems will be graded. Please write legibly.

1. This problem deals with a simplified version of the game Chutes and Ladders. Start with a linear digraph, with vertices labeled $0, 1, ..., n$:

   $$0 \quad \rightarrow \quad 1 \quad \rightarrow \quad 2 \quad \rightarrow \quad \cdots \quad \rightarrow \quad n$$

   A subset of additonal edges i to j where $i + 1 < j$ may be present. Assume for simplicity that each node's out degree is ≤ 2.

   The game starts with a token on node 0. A step of the game means the following: if the token is on node i, randomly choose a successor of i (with equal probability), then move the token there. The game finishes when the token reaches node n.

   Give a polynomial time algorithm to determine the expected duration of the game. Try to make your algorithm as efficient as you can.

2. You are given a list of n intervals on the number line defined by their end points. We say that an interval $[x, y]$ contains another interval $[x', y']$ if $x \leq x' \leq y' \leq y$. Develop an $O(n \log n)$ time algorithm to determine the number of pairs of intervals in the list where one contains the other.

3. Consider the following random process for generating a binary tree. At step 1, the tree contains exactly one node, which is designated the root. At step i, we follow a random path in the tree from the root to a leaf by flipping a coin and going left or right with equal probability until we reach a leaf. Then we add two children to that leaf node. Observe that every node in the tree has either 0 or 2 children, and every step increases the number of leaves by 1.

   Prove that after n steps, the average depth of a leaf is $O(\log n)$ in expectation over the randomness in the process.

(continued on next page.)

4. A Permutation Branching Program (PBP for short) of width w and length L on Boolean variables $X = \{x_1, x_2, \ldots, x_n\}$ is a directed layered graph G with vertex set $V = \{(i, \ell) \mid 1 \le i \le w, 0 \le \ell \le L\}$, in which each layer $V_\ell = \{(i, \ell) \mid 1 \le i \le w\}$ is labeled by some variable x_j. Each layer V_ℓ $(0 \le \ell < L)$ has w edges labeled by 0 and w edges labeled by 1, all going to $V_{\ell+1}$. The edges labeled by 0 prescribe a permutation $\pi_{0,\ell}$ (of $\{1, \ldots, w\}$), and those labeled by 1 prescribe a permutation $\pi_{1,\ell}$ (of $\{1, \ldots, w\}$).

   For any assignment to X, a PBP starts at $(1, 0)$, and follows the path consistent with the assignment, and arrives at some node in layer L. We say a Boolean function $f(X)$ is computed by the PBP if it ends in $(1, L)$ iff $f(X)$ is true.

   - (a.) We can write permutations in cycle notation, e.g. (14325) moves 1 to 4, 4 to 3, etc. We multiply permutations going left to right, for example (14235)(12345) moves 1 to 5 via 4, 4 to 3 via 2, etc. The inverse a^{-1} denotes the inverse permutation of a. Thus e.g., $(14235)^{-1} = (53241) = (15324)$.
     The following identity holds:

     $$aba^{-1}b^{-1} = (14235)(12345)(53241)(54321) = (12534).$$

     Use this to design a width 5 PBP of length 4 that computes the Boolean OR function $f = x \vee y$.
   - (b.) Let TREE be the Boolean function that is represented by a depth d balanced binary tree that alternates OR and AND. The tree has $n = 2^d$ leaves labeled by variables $x_1, x_2, \ldots, x_n$. Give a width 5 PBP that computes TREE.
   - (c.) The following identities also holds:

     $$abba^{-1}b^{-1}b^{-1} = (14235)(13524)(53241)(42531) = (132).$$

     Use this idea to design a width 5 PBP that computes the Boolean function TREE that has length $O(n^c)$ for some $c < 2$.

Good Luck!