Theory Qualifying Exam

Fall 2015

Please answer all four questions below.

1. In the *Balanced Path* problem, we are given a directed graph G with special nodes s and t, and a partition of the edges of the graph into k groups. The goal is to determine whether there is a path from s to t that contains at most one edge of any one group. Prove that the Balanced Path problem is NP-complete.

2. Suppose that you possess a black-box for solving instances of the Knapsack problem exactly. You want to use this black-box for solving the following *Multiple Knapsack* problem: you are given n items of values v_i and weights w_i respectively, as well as k knapsacks, each of capacity 1. Your goal is to find a collection of disjoint subsets of items $S_1, \cdots, S_k$, such that each subset fits into a single knapsack, that is, for all $j \in [k]$, where $[k] = \{1, 2, \ldots, k\}$, $\sum_{i \in S_j} w_i \leq 1$, and the total value $\sum_{i \in \cup_{j \in [k]} S_j} v_i$ is maximized.

   Consider the following greedy algorithm: (1) Use the single-Knapsack black-box to find the optimal subset of items for the first knapsack. (2) Recurse for the remaining knapsacks over the remaining items.

   (a) Prove that the greedy algorithm does not always return the optimal solution.

   (b) Prove that the greedy algorithm obtains a factor of 3/2 approximation for $k = 2$.

3. A branching program (BP) on Boolean variables $\{x_1, \ldots, x_n\}$ is defined as follows. It is a directed graph where vertices are partitioned into layers L_i ($1 \leq i \leq m$), for some m. For each $1 \leq i < m$, there exists $1 \leq j \leq n$, such that L_i is labeled by x_j, and every node in L_i has two edges to nodes in L_{i+1} labeled 0 and 1 respectively. One node in L_1 is the starting node a, and one node in L_m is the accepting node z. Any assignment $\sigma : \{x_1, \ldots, x_n\} \to \{0, 1\}$ defines a unique path from a to some node in L_m by following the edges consistent with σ. The BP is said to compute a Boolean function $f(x_1, \ldots, x_n)$ if: σ defines a path from a to z iff $f(\sigma(x_1), \ldots, \sigma(x_n)) = 1$.

   Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in (\mathbb{Z}_2)^{3\times 3}$ be two matrices over the finite field $\mathbb{Z}_2$. Note that $ABAB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   Build a BP where each node corresponds to a matrix of the form $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, where $(a, b, c) \in \{0, 1\}^3$, to compute the following sequence of functions:

   - $f_1 = x_1 \wedge x_2$, and $g_1 = x_1 \vee x_2$.
   - $f_2 = (x_1 \vee x_2) \wedge (x_3 \vee x_4)$, and $g_2 = (x_1 \wedge x_2) \vee (x_3 \wedge x_4)$.

- $f_k = g_{k-1}(x_1, \ldots, x_{2^{k-1}}) \wedge g_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})$, and $g_k = f_{k-1}(x_1, \ldots, x_{2^{k-1}}) \vee f_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})$.

Your BP should have polynomial length in the number of variables 2^k for f_k and g_k.

4. Consider the following statement: For every $L \in \mathrm{NL}$ there exists a constant c such that for every positive integer d, L can be decided by circuits of depth d and size $2^{n^{c/d}}$ for almost all input lengths n. Show that the statement:

   (a) fails for circuits of bounded fan-in, and

   (b) holds for circuits of unbounded fan-in.

Good Luck!!